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FREE CONVECTION OSCILLATORY FLOW FROM A HORIZONTAL PLATE

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Abstract—Unsteady free convection from a semi-infinite horizontal plate is analysed when the plate temperature varies periodically in time about a constant mean. Separate solutions for low and high frequency ranges are developed. It is found that for low frequencies the oscillatory component of the Nusselt number at the plate increases as the Prandtl number increases. For very high frequencies the temperature field is of the shear-wave type unaffected by the steady mean flow.

NOMENCLATURE

- $\bar{t}$, time;
- t, dimensionless time, $\nu\bar{t}/L_2$;
- $\bar{x}$, co-ordinate along the plate measured from the leading edge;
- x, dimensionless co-ordinate along the plate, $\bar{x}/L$;
- $\bar{y}$, co-ordinate perpendicular to the plate;
- y, dimensionless co-ordinate perpendicular to the plate, $\bar{y}/L$;
- $\bar{u}$, velocity component along $\bar{x}$;
- u, dimensionless velocity component, $\bar{u}L/\nu$;
- $\bar{v}$, velocity component along $\bar{y}$;
- v, dimensionless velocity component, $\bar{v}L/\nu$;
- $\bar{T}$, fluid temperature;
- T, dimensionless fluid temperature, $(\bar{T} - \bar{T}_\infty)/(\bar{T}_w - \bar{T}_\infty)$;
- $\bar{T}_\infty$, free stream temperature;
- $\bar{T}_w$, temperature at the plate;
- L, characteristic length, $[g\beta(\bar{T}_w - \bar{T}_\infty)/\nu^2]^{-\frac{1}{3}}$;
- $\bar{p}$, pressure;
- ρ, density;
- ρ_∞, free stream density;
- β, coefficient of thermal expansion;
- g, acceleration due to gravity;
- k, thermal diffusivity of fluid, $\kappa/\rho c_p$;
- κ, thermal conductivity of fluid;
- c_p, specific heat of fluid;
- μ, fluid viscosity;
- ν, kinematic viscosity of fluid, μ/ρ;
- σ, Prandtl number, ν/K;
- $\bar{\omega}$, frequency of oscillation;
- ω, dimensionless frequency, $\bar{\omega}L^2/\nu$;
- ϵ, amplitude of oscillation;
- N, a constant depending on σ, $[\frac{1}{2}(7/17\sigma)^{\frac{1}{5}}\, 10^{\frac{4}{5}}]$;
- u_s, v_s, T_s, steady flow components;
- u_1, v_1, T_1, small oscillating components;
- $\delta(x)$, dimensionless boundary-layer thickness;
- q_0, local heat transfer at the plate in steady state;
- Nu_0, local Nusselt number in the steady state;
- τ_0, local skin friction;
- τ_0^*, non-dimensional local skin friction at the plate;
- u_r, v_r, T_r, in-phase components;
- u_2, v_2, T_2, out-of-phase components;
- T_0, temperature difference between the plate and free-stream temperatures, $\bar{T}_w - \bar{T}_\infty$;
- $\bar{u}_s, \bar{v}_s, \bar{T}_s$, steady dimensional mean flow components;
- Nu, local Nusselt number;
- q, local heat transfer at the plate for

	small frequency oscillation;
Nu_1,	small oscillating Nusselt number;
τ,	skin friction;
τ^*,	non-dimensional skin friction, $\tau_0^* + \epsilon\tau_1^*$;
τ_1^*,	non-dimensional small oscillating component of skin friction;
η,	non-dimensional variable, $y\sqrt{\omega}$.

1. INTRODUCTION

THE PRESENT paper is devoted to a study of the flow and heat transfer from a semi-infinite horizontal plate whose temperature oscillates about a constant mean. Free convection flow from a horizontal plate has not received much attention. Recently Gill and Casal [1] considered some aspects of this problem. They obtained similarity solutions of the boundary-layer equations for steady flow over a semi-infinite horizontal plate. Sparrow and Minkowycz [2] have also considered free convection effects on a horizontal plate by employing a series expansion of the stream function, which gives the perturbation of a basic forced convection flow due to buoyancy. The basic steady flow in our case is different since we have assumed the free stream to be at rest. Thus the basic flow is entirely due to buoyancy forces over a horizontal plate whose temperature differs from that of the free stream. The effect of the buoyancy forces is to induce a longitudinal pressure gradient which causes flow. It is an interesting flow in its own right yielding a steady outer flow for the boundary-layer equations as a result of free convection alone. Moreover this problem should be easily amenable to experiment in a laboratory. We have, therefore, considered the basic steady flow also in detail since there is no previous theoretical work reported in the literature on this problem. For the situation in which the plate temperature $\bar{T}_w$ is greater than the free stream temperature $\bar{T}_\infty$, it will be difficult to realise such a flow above a horizontal plate because of the unstable stratification due to the piling of "heavier fluid on lighter fluid". In such cases one should consider the flow below the plate.

The study of the oscillatory flow is restricted to small amplitudes (ϵ) only. This enables us to employ the techniques of linearization and still retain first-order effects of plate temperature fluctuations. This type of technique was first employed by Lighthill [4] and later used by many workers. Separate solutions for low and high frequencies are developed. For low frequencies and for $\epsilon = 0{\cdot}1$, the analysis predicts a 12 per cent increase in the Nusselt number and about 6 per cent increase in the skin friction coefficient at the plate. In the limiting case of very high frequencies, the oscillatory component of the temperature field is of the simple shear wave type unaffected by the mean flow predicting a phase lead of $\pi/4$ in the heat transfer at the plate. In the intermediate frequency range the velocity and temperature fields are more complicated due to the interaction of the steady mean flow.

2. BASIC EQUATIONS

Consider a semi-infinite horizontal flat plate with x-axis along the plate measured from the leading edge and y-axis vertically upwards. The usual boundary-layer equations in two dimensional flow are

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \quad (2.1)$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}} - g\rho, \quad (2.2)$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad (2.3)$$

$$\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = K\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \quad (2.4)$$

where g is the acceleration due to gravity and $K(=\kappa/\rho c_p)$ is the thermal diffusivity.

In accordance with the usual practice, we shall consider density variations only in the buoyancy force, other density variations will be neglected within the framework of incompressible fluids. The simplest way to do so is to take the equation of state in the form

$$\rho = \rho_\infty[1 - \beta(\bar{T} - \bar{T}_\infty)], \tag{2.5}$$

where β is the coefficient of thermal expansion. ρ_∞ is the free stream density and $\bar{T}_\infty$ is the free stream temperature.

It is convenient to eliminate $\bar{p}$ from (2.1) and (2.2) by differentiating (2.1) with respect to $\bar{y}$ and (2.2) with respect to $\bar{x}$ and subtracting and making use of (2.5) to get

$$\frac{\partial}{\partial \bar{y}}\left[\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + v\frac{\partial \bar{u}}{\partial \bar{y}}\right] = v\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} - g\beta\frac{\partial}{\partial \bar{x}}(\bar{T} - \bar{T}_\infty), \tag{2.6}$$

Introducing the dimensionless variables

$$\left.\begin{aligned} x = \bar{x}/L, \quad y = \bar{y}/L, \quad t = v\bar{t}/L^2, \quad u = \bar{u}L/v, \quad v = \bar{v}L/v,\\ T = (\bar{T} - \bar{T}_\infty)/(\bar{T}_w - \bar{T}_\infty), \end{aligned}\right\} \tag{2.7}$$

where L is the characteristic length $[g\beta(\bar{T}_w - \bar{T}_\infty)/v^2]^{-\frac{1}{3}}$ and $\bar{T}_w(> \bar{T}_\infty)$ is the mean temperature of the plate, equations (2.6), (2.3) and 2.4) finally take the forms

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \frac{\partial^3 u}{\partial y^3} - \frac{\partial T}{\partial x}, \tag{2.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.9}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{2.10}$$

where σ is the Prandtl number.

It may be remarked here that these equations do not hold at the leading edge. Therefore, it is probable that the results may be expected to apply only far from the leading edge.

We shall consider the case when the plate temperature oscillates harmonically in time about a non-zero mean. The boundary conditions to be satisfied will be

$$\begin{aligned} &u = 0, \quad v = 0, \quad T = 1 + \epsilon\cos\omega t, \quad \epsilon \ll 1 \quad &\text{at} \quad y = 0,\\ &u \to 0, \quad T \to 0 &\text{as} \quad y \to \infty, \end{aligned} \tag{2.11}$$

where ω is the dimensionless frequency $\bar{\omega}L^2/v$.

3. METHOD OF SOLUTION

In order to solve the above differential equations it is convenient to use the complex notation for harmonic functions in which only real parts will have physical meaning. The plate temperature, which can be written as $[\bar{T}_w + \epsilon(\bar{T}_w - \bar{T}_\infty)\exp(i\bar{\omega}\bar{t})]$, consists of a basic steady distribution $\bar{T}_w$ with a superimposed weak time varying distribution $\epsilon(\bar{T}_w - \bar{T}_\infty)\exp(i\bar{\omega}\bar{t})$.

We now write u, v and T as the sum of steady and small oscillating components

$$\left.\begin{aligned} u &= u_s + \epsilon u_1 \exp(i\omega t),\\ v &= v_s + \epsilon v_1 \exp(i\omega t),\\ T &= T_s + \epsilon T_1 \exp(i\omega t), \end{aligned}\right\} \tag{3.1}$$

where u_s, v_s, T_s is the steady mean flow and satisfies

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left[u_s\frac{\partial u_s}{\partial x}+v_s\frac{\partial u_s}{\partial y}\right] &= \frac{\partial^3 u_s}{\partial y^3}-\frac{\partial T_s}{\partial x}, \\ \frac{\partial u_s}{\partial x}+\frac{\partial v_s}{\partial y} &= 0, \\ u_s\frac{\partial T_s}{\partial x}+v_s\frac{\partial T_s}{\partial y} &= \frac{1}{\sigma}\frac{\partial^2 T_s}{\partial y^2}, \end{aligned}\right\} \tag{3.2}$$

with the boundary conditions

$$\left.\begin{aligned} u_s = v_s = 0, \quad T_s = 1 \quad &\text{at} \quad y = 0, \\ u_s \to 0, \quad T_s \to 0 \quad &\text{as} \quad y \to \infty. \end{aligned}\right\} \tag{3.3}$$

Neglecting squares of ϵ and dividing by exp $(i\omega t)$, we find that u_1, v_1, T_1 satisfy the following differential set,

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left[i\omega u_1+u_1\frac{\partial u_s}{\partial x}+u_s\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_s}{\partial y}+v_s\frac{\partial u_1}{\partial y}\right] = \frac{\partial^3 u_1}{\partial y^3}-\frac{\partial T_1}{\partial x}, \qquad \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y} = 0, \\ i\omega T_1+u_1\frac{\partial T_s}{\partial x}+u_s\frac{\partial T_1}{\partial x}+v_1\frac{\partial T_s}{\partial y}+v_s\frac{\partial T_1}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 T_1}{\partial y^2}, \end{aligned}\right\} \tag{3.4}$$

with the boundary conditions

$$\left.\begin{aligned} u_1 = v_1 = 0, \quad T_1 = 1 \quad &\text{at} \quad y = 0, \\ u_1 \to 0, \quad T_1 \to 0 \quad &\text{as} \quad y \to \infty. \end{aligned}\right\} \tag{3.5}$$

Equations (3.2) and (3.3) are the boundary-layer equations which describe the steady state free convection flow over a semi-infinite horizontal flat plate maintained at a constant temperature $\bar{T}_w$. These equations are the same as those obtained by Gill and Casal [1] but the boundary conditions are different. They have assumed the existence of a uniform free stream velocity whereas in our case the free stream is assumed to be at rest. Thus the flow is purely due to buoyancy effects which induce a longitudinal pressure gradient given by

$$\frac{\partial \bar{p}}{\partial x} = -g\beta\rho_\infty \int_{\bar{y}}^{\infty} \frac{\partial}{\partial \bar{x}}(\bar{T}-\bar{T}_\infty)\,\mathrm{d}\bar{y}, \tag{3.6}$$

which causes the flow.

For flow below the plate, the co-ordinate y would be reversed to measure distances vertically downwards and the negative sign in equation (3.6) would be deleted. It follows with a flow above the plate for which $\bar{T}_w > \bar{T}_\infty$ or for a flow below the plate for which $\bar{T}_w < \bar{T}_\infty$, the induced pressure gradient $\partial\bar{p}/\partial\bar{x}$ is negative resulting in an accelerated flow. For flow below the plate for $\bar{T}_w > \bar{T}_\infty$ or for flow above the plate for $\bar{T}_w < \bar{T}_\infty$, the situation will be just reversed. It is therefore sufficient to consider only one of the four situations. For convenience we shall discuss flow above the plate for $\bar{T}_w > \bar{T}_\infty$.

We shall first consider the set of equations which describe the basic steady flow. We shall integrate these equations by Kármán–Pohlhausen method since our aim is to get qualitative results. Towards this end we put these set of equations in integro-differential form by integrating from $y = 0$ to $y = \delta$, where $\delta(x)$ is the dimensionless boundary-layer thickness, as

$$\left(\frac{\partial^2 u_s}{\partial y^2}\right)_{y=0} + \frac{\partial}{\partial x}\int_0^\delta T_s \,\mathrm{d}y = 0, \tag{3.7}$$

$$\frac{\partial}{\partial x}\int_0^\delta u_s T_s \,\mathrm{d}y + \frac{1}{\sigma}\left(\frac{\partial T_s}{\partial y}\right)_{y=0} = 0. \tag{3.8}$$

Consistent with the boundary conditions

$$\left.\begin{aligned} &u_s = 0, \quad T_s = 1 && \text{at} \quad y = 0, \\ &u_s, \quad \frac{\partial u_s}{\partial y}, \quad \frac{\partial^2 u_s}{\partial y^2} \to 0, \quad T_s \to 0 && \text{as} \quad y \to \delta, \end{aligned}\right\} \tag{3.9}$$

we take the expressions for u_s and T_s as

$$u_s = A\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^3\left(1 + \frac{y}{\delta}\right), \tag{3.10}$$

$$T_s = \left(1 + \frac{y}{\delta}\right)\left(1 - \frac{y}{\delta}\right)^3, \tag{3.11}$$

where A and δ are functions of x to be determined from equations (3.7) and (3.8). These expressions for u_s and T_s also satisfy the additional conditions

$$\frac{\partial^3 u_s}{\partial y^3} = 0, \quad \frac{\partial^2 T_s}{\partial y^2} = 0 \qquad \text{at} \quad y = 0, \tag{3.12}$$

which are obtained by evaluating the first and third equations of (3.2) at $y = 0$.

Substituting these expressions (3.10) and (3.11) into (3.7) and (3.8), we get

$$\frac{4A}{\delta^2} - \frac{3}{10}\frac{\mathrm{d}}{\mathrm{d}x}(\delta) = 0, \tag{3.13}$$

$$\frac{17}{630}\frac{\mathrm{d}}{\mathrm{d}x}(A\delta) - \frac{2}{\sigma\delta} = 0. \tag{3.14}$$

The solutions of these equations are easily found to be

$$A = \frac{6}{25}N^3 x^{\frac{1}{5}}, \tag{3.15}$$

$$\delta = 2Nx^{\frac{2}{5}}, \tag{3.16}$$

where

$$N = \left[\tfrac{1}{2}\left(\frac{7}{17\sigma}\right)^{\frac{1}{5}} 10^{\frac{4}{5}}\right].$$

From these simple expressions we find that the boundary-layer thickness varies as $x^{\frac{2}{5}}$ and decreases as the Prandtl number increases. It is also clear from (3.16) that $(y/2x^{\frac{2}{5}})$ is the similarity variable for free convection from a horizontal plate. The Table 1 shows the values of $(\delta/2x^{\frac{2}{5}})$ for various values of σ.

In Figs. 1–3 are plotted $u_s/x^{\frac{1}{5}}$ and T_s against $y/2x^{\frac{2}{5}}$ for a fairly representative range of values of σ. It is found that both the velocity and the temperature at any point within the boundary layer decrease as the Prandtl number increases.

The local heat-transfer at the plate is given by

$$q_0 = -\frac{K\nu^2}{g\beta L^4}\left(\frac{\partial T_s}{\partial y}\right)_{y=0} = \frac{K\nu^2}{g\beta L^4}\frac{x^{-\frac{2}{5}}}{N},$$

Table 1

	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}1$	$\sigma = 0{\cdot}72$	$\sigma = 1$	$\sigma = 5$	$\sigma = 10$
$\delta/2x^{\frac{2}{5}}$	6·6349	4·1867	2·8215	2·6416	1·9146	1·6667

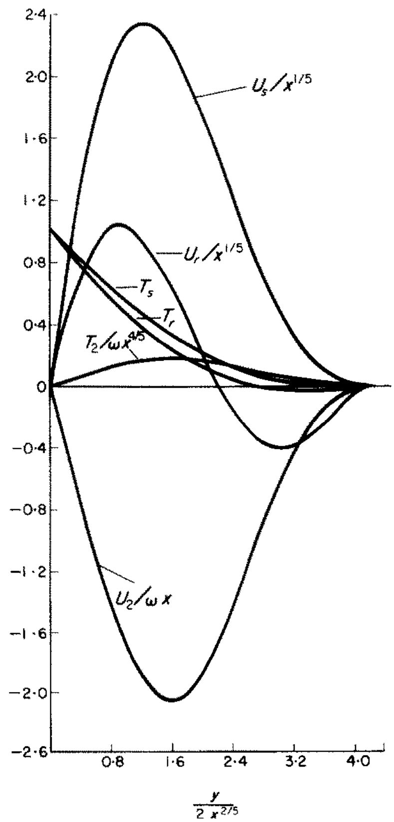

FIG. 1. The steady and oscillating components of velocity and temperature for $\sigma = 0{\cdot}1$.

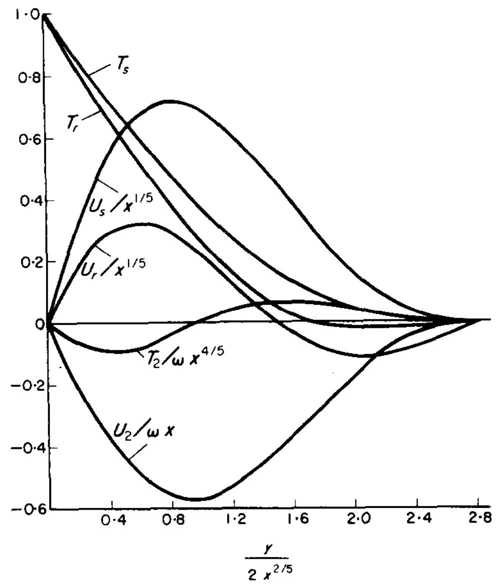

FIG. 2. The steady and oscillating components of velocity and temperature for $\sigma = 0{\cdot}72$.

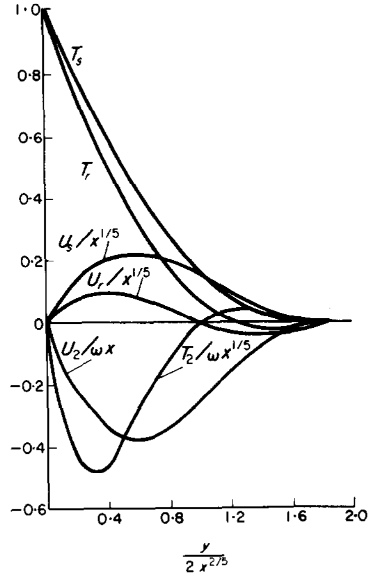

FIG. 3. The steady and oscillating components of velocity and temperature for $\sigma = 5$.

which can be expressed in terms of the Nusselt number as

$$Nu_0 = \frac{q_0 L}{K(\bar{T}_w - \bar{T}_\infty)} = \frac{1}{N} x^{-\frac{2}{5}}. \tag{3.17}$$

Similarly the skin friction at the plate can be expressed as

$$\tau_0^* = \frac{\tau_0 L^2}{\rho \nu^2} = \frac{3}{25} N^2 x^{-\frac{1}{5}}. \tag{3.18}$$

The expressions for $Nu_0\, x^{\frac{2}{5}}$ and $\tau_0^*\, x^{\frac{1}{5}}$ are calculated for various values of σ and given in Table 3. The table shows that the Nusselt number at the plate increases while the skin friction decreases as σ increases.

Set (3.4) is next considered. It is convenient to write u_1, v_1 and T_1 as the sum of in-phase and out-of-phase components as

$$\left.\begin{aligned} u_1 &= u_r + iu_2, \\ v_1 &= v_r + iv_2, \\ T_1 &= T_r + iT_2. \end{aligned}\right\} \tag{3.19}$$

Substituting into (3.4) and separating real and imaginary parts we get

$$\frac{\partial}{\partial y}\left[-\,\omega u_2 + u_s\frac{\partial u_r}{\partial x} + v_s\frac{\partial u_r}{\partial y} + u_r\frac{\partial u_s}{\partial x} + v_r\frac{\partial u_s}{\partial y}\right] = \frac{\partial^3 u_r}{\partial y^3} - \frac{\partial T_r}{\partial x}, \tag{3.20}$$

$$\frac{\partial u_r}{\partial x} + \frac{\partial v_r}{\partial y} = 0, \tag{3.21}$$

$$-\,\omega T_2 + u_s\frac{\partial T_r}{\partial x} + v_s\frac{\partial T_r}{\partial y} + u_r\frac{\partial T_s}{\partial x} + v_r\frac{\partial T_s}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 T_r}{\partial y^2} \tag{3.22}$$

with the boundary conditions

$$\left.\begin{array}{lll} u_r = v_r = 0, & T_r = 1 & \text{at} \quad y = 0, \\ u_r \to 0, & T_r \to 0 & \text{as} \quad y \to \infty, \end{array}\right\} \tag{3.23}$$

and

$$\frac{\partial}{\partial y}\left[\omega u_r + u_s \frac{\partial u_2}{\partial x} + v_s \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_s}{\partial x} + v_2 \frac{\partial u_s}{\partial y}\right] = \frac{\partial^3 u_2}{\partial y^3} - \frac{\partial T_2}{\partial x}, \tag{3.24}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \tag{3.25}$$

$$\omega T_r + u_s \frac{\partial T_2}{\partial x} + v_s \frac{\partial T_2}{\partial y} + u_2 \frac{\partial T_s}{\partial x} + v_2 \frac{\partial T_s}{\partial y} = \frac{1}{\sigma} \frac{\partial^2 T_2}{\partial y^2} \tag{3.26}$$

with the boundary conditions

$$\begin{array}{lll} u_2 = v_2 = 0, & T_2 = 0 & \text{at} \quad y = 0, \\ u_2 \to 0, & T_2 \to 0 & \text{as} \quad y \to \infty. \end{array} \tag{3.27}$$

Thus

$$\tan^{-1}\left(\frac{u_2}{u_r}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{T_2}{T_r}\right)$$

will represent the phase shifts in the velocity and temperature fluctuations in the boundary layer due to fluctuations in the plate temperature. When the frequency of oscillation is low it is to be expected that the phase shifts will be small. Therefore, one would expect u_2 and T_2 to be small relative to u_r and T_r. Thus when ω is small, the terms $(-\omega u_2)$ and $(-\omega T_2)$ can be neglected in (3.20) and (3.22). u_r, v_r and T_r will then be the quasi-steady solution corresponding to $\omega = 0$. This can also be seen from the fact that the same equations can be obtained by substituting $u = u_s + \epsilon u_r$, $v = v_s + \epsilon v_r$ and $T = T_s + \epsilon T_r$ in the steady flow boundary-layer equations. It can be easily verified that

$$\left.\begin{array}{l} u_r = \dfrac{LT_0}{\nu}\left(\dfrac{\partial \bar{u}_s}{\partial T_0}\right), \\ v_r = \dfrac{LT_0}{\nu}\left(\dfrac{\partial \bar{v}_s}{\partial T_0}\right), \\ T_r = \dfrac{\partial \bar{T}_s}{\partial T_0}, \end{array}\right\} \tag{3.28}$$

where $T_0 = (\bar{T}_w - \bar{T}_\infty)$. With the help of equations (3.10) and (3.11) we have

$$u_r = \frac{6}{125} N^3 \left[\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \left(3 - 2\frac{y}{\delta} - 7\frac{y^2}{\delta^2}\right)\right] x^{\frac{1}{5}}, \tag{3.29}$$

$$T_r = \frac{1}{5} \left[\left(1 - \frac{y}{\delta}\right)^2 \left(5 - 2\frac{y}{\delta} - 9\frac{y^2}{\delta^2}\right)\right] \tag{3.30}$$

and v_r can be obtained from the equation of continuity (3.21).

We shall next consider equations (3.24) and (3.26) through (3.27). Accordingly we assume polynomials for u_2 and T_2 as

$$u_2 = A_0 + A_1\frac{y}{\delta} + A_2\frac{y^2}{\delta^2} + A_3\frac{y^3}{\delta^3} + A_4\frac{y^4}{\delta^4} + A_5\frac{y^5}{\delta^5}, \tag{3.31}$$

$$T_2 = C_0 + C_1\frac{y}{\delta} + C_2\frac{y^2}{\delta^2} + C_3\frac{y^3}{\delta^3} + C_4\frac{y^4}{\delta^4} + C_5\frac{y^5}{\delta^5}, \tag{3.32}$$

where A's and C's are functions of x only. These will be determined by imposing the following boundary conditions on u_2 and T_2 and their derivatives:

$$\left.\begin{aligned} &u_2 = 0, \quad T_2 = 0 && \text{at} \quad y = 0,\\ &u_2 = \frac{\partial u_2}{\partial y} = \frac{\partial^2 u_2}{\partial y^2} = 0, \quad T_2 = \frac{\partial T_2}{\partial y} = \frac{\partial^2 T_2}{\partial y^2} = 0 && \text{at} \quad y = \delta. \end{aligned}\right\} \tag{3.33a}$$

Two more boundary conditions can be generated by evaluating equations (3.24) and (3.26) at $y = 0$ as

$$\left.\begin{aligned} \omega\left(\frac{\partial u_r}{\partial y}\right)_{y=0} &= \left(\frac{\partial^3 u_2}{\partial y^3}\right)_{y=0},\\ \omega(T_r)_{y=0} &= \frac{1}{\sigma}\left(\frac{\partial^2 T_2}{\partial y^2}\right)_{y=0}. \end{aligned}\right\} \tag{3.33b}$$

Equations (3.31) and (3.32) finally take the forms

$$u_2 = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^3\left[\frac{\omega A\delta^2}{30} - A_5\left(1 + \frac{y}{\delta}\right)\right], \tag{3.34}$$

$$T_2 = -\frac{1}{3}\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^3\left[\frac{1}{2}\omega\sigma\delta^2 + C_5\left(1 + 3\frac{y}{\delta}\right)\right]. \tag{3.35}$$

Here A_5 and C_5 are still undetermined. They will be determined from the momentum integral equations corresponding to equations (3.24) and (3.26). Integrating equations (3.24) and (3.26) from $y = 0$ to $y = \delta$ and making use of (3.25) and (3.27) we get the momentum integral equations as

$$\frac{\partial}{\partial x}\int_0^\delta T_2\,\mathrm{d}y + \left(\frac{\partial^2 u_2}{\partial y^2}\right)_{y=0} = 0, \tag{3.36}$$

$$\omega\int_0^\delta T_r\,\mathrm{d}y + \frac{\partial}{\partial x}\int_0^\delta (u_s T_2 + u_2 T_s)\,\mathrm{d}y + \frac{1}{\sigma}\left(\frac{\partial T_2}{\partial y}\right)_{y=0} = 0. \tag{3.37}$$

Substituting equations (3.34) and (3.35) into (3.36) and (3.37) we get

$$\frac{4A_5}{\delta^2} - \frac{\omega A}{5} - \frac{1}{30}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\omega\sigma\delta^3}{4} + \delta C_5\right] = 0, \tag{3.38}$$

$$\frac{6}{25}\omega\delta + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{11}{15120}\omega A\delta^3 - \frac{17}{630}\delta A_5 - \frac{13}{15120}A\omega\sigma\delta^3 - \frac{139}{41580}A\delta C_5\right] - \frac{\omega\delta}{6} - \frac{C_5}{3\sigma\delta} = 0. \tag{3.39}$$

These differential equations can be easily solved to give

$$\left.\begin{aligned} A_5 &= \omega F x, \\ C_5 &= \omega E x^{\frac{4}{5}}, \end{aligned}\right\} \tag{3.40}$$

where E and F are constants given by

$$125F - 10N^3E - (6 + 10\sigma)N^5 = 0$$

and

$$\frac{11}{75}N + \frac{N^6}{5625}(11 - 13\sigma) - \frac{17}{225}NF - \frac{139}{61875}EN^4 - \frac{E}{6\sigma N} = 0. \tag{3.41}$$

The Table 2 shows the values of E and F for various values of σ.

Table 2

	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}1$	$\sigma = 0{\cdot}72$	$\sigma = 1$	$\sigma = 5$	$\sigma = 10$
E	−7·9945	−4·3896	−5·6952	−6·4591	−14·3936	−21·3489
F	440·4757	46·3781	8·6489	6·9392	3·4410	2·9989

4. RESULTS

When the frequency of oscillation is small, the temperature in the boundary layer and the longitudinal component of velocity may be written in the forms

$$\left.\begin{aligned} T &= T_s + \epsilon R_1 \cos(\omega t + \alpha_1), \\ u &= u_s + \epsilon R_2 \cos(\omega t + \alpha_2), \end{aligned}\right\} \tag{4.1}$$

where

$$R_1 = (T_r^2 + T_2^2)^{\frac{1}{2}}, \qquad R_2 = (u_r^2 + u_2^2)^{\frac{1}{2}},$$

$$\alpha_1 = \tan^{-1}(T_2/T_r), \qquad \alpha_2 = \tan^{-1}(u_2/u_r).$$

The functions u_r, u_2, T_r and T_2 are shown in Figs 1–3 for $\sigma = 0{\cdot}1$, $0{\cdot}72$ and 5. u_r is positive near the plate and negative away from the plate. For the values of σ considered it is found that the lower is the Prandtl number, greater is the distance from the plate where u_r changes its sign. On the other hand, u_2 is always negative and increases in magnitude as the Prandtl number decreases. This shows that the velocity fluctuations near the plate lag behind the plate temperature oscillations. On the other hand for $\sigma = 0{\cdot}1$, T_2 is positive near the plate but for $\sigma = 0{\cdot}72$ and $\sigma = 5$, it is negative but T_r remains positive. This shows that the temperature oscillations near the plate anticipate the plate temperature oscillations for small values of σ. This phase lead is more than compensated as σ increases.

The local Nusselt number at the plate can be expressed as

$$Nu = \frac{qL}{K(\bar{T}_w - \bar{T}_\infty)} = \left[\frac{1}{N\,x^{\frac{2}{5}}} + \epsilon\, R_3 \cos(\omega t + \alpha_3)\right] = Nu_0 + \epsilon\, Nu_1, \tag{4.2}$$

where

$$R_3 = \left[\left(\frac{6}{5N\,x^{\frac{2}{5}}}\right)^2 + \frac{\omega^2 x^{\frac{4}{5}}}{36N^2}\left(2\sigma N^2 + E\right)^2\right]^{\frac{1}{2}},$$

$$\alpha_3 = \tan^{-1}\left[\frac{5\omega\, x^{\frac{4}{5}}}{36}(2\sigma N^2 + E)\right].$$

The local skin-friction at the plate can be calculated as

$$\tau^* = \frac{\tau L^2}{\rho v^2} = \left(\frac{\partial u_s}{\partial y}\right)_{y=0} + \epsilon \exp(i\omega t)\left(\frac{\partial u_1}{\partial y}\right)_{y=0}$$

$$= \frac{3}{25}N^2 x^{-\frac{1}{5}} + \epsilon\, R_4 \cos(\omega t + \alpha_4)$$

$$= \tau_0^* + \epsilon\, \tau_1^* \tag{4.3}$$

where

$$R_4 = \left[\left(\frac{9}{125}\frac{N^2}{x^{\frac{1}{5}}}\right)^2 + \frac{\omega^2 x^{\frac{6}{5}}}{4N^2}\left(\frac{4}{125}N^5 - F\right)^2\right]^{\frac{1}{2}},$$

$$\alpha_4 = \tan^{-1}\left[\frac{125}{18}\frac{\omega x^{\frac{4}{5}}}{N^3}\left(\frac{4}{125}N^5 - F\right)\right].$$

This shows that both $\tan\alpha_3$ and $\tan\alpha_4$ vary as $\omega x^{\frac{4}{5}}$. In Figs. 4 and 5 are plotted the functions $(\tan\alpha_3/\omega x^{\frac{4}{5}})$ and $(\tan\alpha_4/\omega x^{\frac{4}{5}})$ against σ. It is clear that $\tan\alpha_3$ is positive for all values of $\sigma > 0{\cdot}12$, but $\tan\alpha_4$ is always negative. This shows that the oscillating component of the Nusselt number at the plate anticipates while the skin friction lags behind the plate temperature oscillations. The amplitudes R_3 and R_4 are found to increase as ω increases. The minimum values of R_3 and $R_4(\omega = 0)$ are given in Table 3 for various σ along with τ_0^* and Nu_0.

It is interesting to note that the ratios $[\epsilon\, Nu_1(\omega = 0)/Nu_0]$ and $[\epsilon\, \tau_1^*(\omega = 0)/\tau_0^*]$ remain constant for all values of σ. Thus for $\epsilon = 0{\cdot}1$ the quasi-steady Nusselt number is 12 per cent of the steady value whereas the skin friction coefficient is 6 per cent of the steady mean value.

High-frequency oscillations

For high-frequencies, Lighthill [3] has shown that the oscillatory flow is of the "shear wave" type and its thickness is small compared to the steady boundary-layer thickness. Within this layer the steady flow may be approximated as

$$
\left.\begin{aligned}
u_s &\approx u_s(0) + y\left(\frac{\partial u_s}{\partial y}\right)_{y=0} + \frac{y^2}{L^2}\left(\frac{\partial^2 u_s}{\partial y^2}\right)_{y=0} + \dots \\
&\approx \frac{3}{25}N^2 x^{-\frac{1}{5}} y - \frac{3}{25} N x^{-\frac{3}{5}} y^2 + \dots, \\
v_s &\approx v_s(0) + y\left(\frac{\partial v_s}{\partial y}\right)_{y=0} + \frac{y^2}{L^2}\left(\frac{\partial^2 v_s}{\partial y^2}\right)_{y=0} + \dots \\
&\approx \frac{3}{125}N^2 x^{-\frac{6}{5}} y^2 + \dots, \\
T_s &\approx T_s(0) + y\left(\frac{\partial T_s}{\partial y}\right)_{y=0} + \frac{y^2}{L_2}\left(\frac{\partial^2 T_s}{\partial y^2}\right)_{y=0} + \dots \\
&\approx 1 - \frac{1}{N x^{\frac{2}{5}}} y + \dots \quad .
\end{aligned}\right\} \quad (4.4)
$$

If we introduce the variable $\eta = y\sqrt{\omega}$ in the equations (3.4) we get

$$
\left.\begin{aligned}
&\frac{\partial^3 u_1}{\partial \eta^3} - i\frac{\partial u_1}{\partial \eta} = \frac{1}{\omega}\frac{\partial}{\partial \eta}\left(u_1\frac{\partial u_s}{\partial x} + u_s\frac{\partial u_1}{\partial x}\right) \\
&\quad + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial \eta}\left(v_1\frac{\partial u_s}{\partial \eta} + v_s\frac{\partial u_1}{\partial \eta}\right) + \frac{1}{\omega^{\frac{3}{2}}}\frac{\partial T_1}{\partial x}, \\
&\frac{\partial u_1}{\partial x} + \sqrt{\omega}\frac{\partial v_1}{\partial \eta} = 0, \\
&\frac{\partial^2 T_1}{\partial \eta^2} - i\sigma T_1 = \frac{\sigma}{\omega}\left(u_1\frac{\partial T_s}{\partial x} + u_s\frac{\partial T_1}{\partial x}\right) + \frac{\sigma}{\sqrt{\omega}}\left(v_1\frac{\partial T_s}{\partial \eta} + v_s\frac{\partial T_1}{\partial \eta}\right).
\end{aligned}\right\} \quad (4.5)
$$

This suggests that for large ω, a solution may be developed in inverse powers of $\sqrt{\omega}$. We write

$$
\left.\begin{aligned}
u_1 &= u_{10} + \frac{1}{\sqrt{\omega}}u_{11} + \frac{1}{\omega}u_{12} + \frac{1}{\omega^{\frac{3}{2}}}u_{13} + \dots, \\
T_1 &= T_{10} + \frac{1}{\sqrt{\omega}}T_{11} + \frac{1}{\omega}T_{12} + \frac{1}{\omega^{\frac{3}{2}}}T_{13} + \dots \quad .
\end{aligned}\right\} \quad (4.6a)
$$

Substituting the series (4.6a) in equation (4.5) and using the approximated steady flow given by (4.4), we have for the first approximation

$$
\left.\begin{aligned}
&\frac{\partial^3 u_{10}}{\partial \eta^3} - i\frac{\partial u_{10}}{\partial \eta} = 0, \\
&\frac{\partial^2 T_{10}}{\partial \eta^2} - i\sigma T_{10} = 0
\end{aligned}\right\} \quad (4.6b)
$$

Table 3

	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}1$	$\sigma = 0{\cdot}72$	$\sigma = 1$	$\sigma = 5$	$\sigma = 10$
$x^{\frac{4}{5}} R_3 (\omega = 0)$	0·1808	0·2866	0·4253	0·4542	0·6268	0·7200
$x^{\frac{1}{5}} R_4 (\omega = 0)$	3·1696	1·2620	0·5732	0·5025	0·2639	0·2000
$x^{\frac{1}{5}} \tau_0^*$	5·2826	2·1034	0·9553	0·8373	0·4399	0·3333
$x^{\frac{4}{5}} Nu_0$	0·1507	0·2388	0·3544	0·3785	0·5223	0·6000
$\dfrac{Nu_1(\omega = 0)}{Nu_0}$	1·20	1·20	1·20	1·20	1·20	1·20
$\dfrac{\tau_1^*(\omega = 0)}{\tau_0^*}$	0·60	0·60	0·60	0·60	0·60	0·60

subject to the boundary conditions

$$\left.\begin{aligned} u_{10} &= 0, \quad T_{10} = 1 \quad \text{at} \quad \eta = 0, \\ u_{10} &\to 0, \quad T_{10} \to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{4.7}$$

The solutions of these equations give

$$\left.\begin{aligned} u_{10} &= 0, \\ T_{10} &= \exp\left[-(i\sigma)^{\frac{1}{2}}\eta\right] \end{aligned}\right\} \tag{4.8}$$

which is unaffected by the steady mean flow. Interaction terms, however, appear in the subsequent higher approximations.

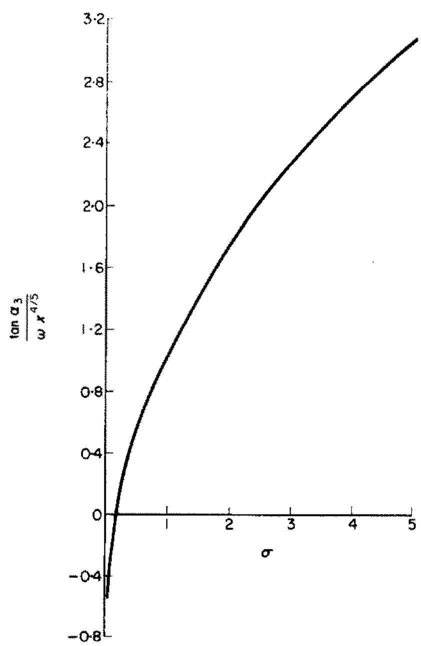

FIG. 4. Tangent of phase angle of the oscillating temperature gradient (low frequency).

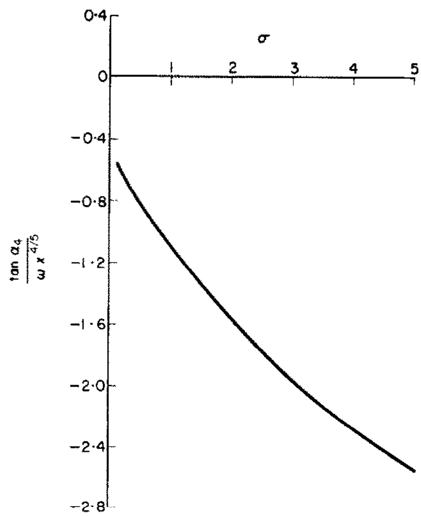

FIG. 5. Tangent of phase angle of the oscillating velocity gradient (low frequency).

For the next approximation we have

$$\left.\begin{aligned}&\frac{\partial^3 u_{11}}{\partial\eta^3} - i\frac{\partial u_{11}}{\partial\eta} = 0,\\&\frac{\partial^2 T_{11}}{\partial\eta^2} - i\sigma T_{11} = 0\end{aligned}\right\} \tag{4.9}$$

subject to the boundary conditions

$$\left.\begin{aligned}&u_{11} = 0, \quad T_{11} = 0 \quad \text{at} \quad \eta = 0,\\&u_{11} \to 0, \quad T_{11} \to 0 \quad \text{as} \quad \eta \to \infty.\end{aligned}\right\} \tag{4.10}$$

The solutions of these equations are evidently

$$u_{11} = 0, \qquad T_{11} = 0.$$

In a similar way T_{12} and u_{12} come out to be zero. In fact the next non-zero term in the expansion of T_1 is T_{13} and it satisfies the equation

$$\frac{\partial^2 T_{13}}{\partial\eta^2} - i\sigma T_{13} = \frac{3}{125} N^2 \sigma x^{-\frac{6}{5}} \eta^2 \frac{\partial T_{10}}{\partial\eta} \tag{4.11}$$

with the boundary conditions

$$\left.\begin{aligned}&T_{13} = 0 \quad \text{at} \quad \eta = 0,\\&T_{13} \to 0 \quad \text{as} \quad \eta \to \infty.\end{aligned}\right\} \tag{4.12}$$

Solving we get,

$$T_{13} = \frac{3}{250}\sigma N^2 x^{-\frac{6}{5}} \left(\frac{1}{3}\eta^3 + \frac{1}{2\sqrt{(i\sigma)}}\eta^2 + \frac{1}{2i\sigma}\eta\right)\exp\left[-(\sqrt{i\sigma})\eta\right]. \tag{4.13}$$

The first non-zero term in the expansion of u_1 is found to be u_{16} and it satisfies the equation

$$\frac{\partial^3 u_{16}}{\partial\eta^3} - i\frac{\partial u_{16}}{\partial\eta} = \frac{\partial T_{13}}{\partial x} \tag{4.14}$$

with the boundary conditions

$$\left.\begin{aligned}&u_{16} = 0 \qquad\qquad\qquad \text{at} \quad \eta = 0,\\&u_{16} \to 0, \quad \frac{\partial u_{16}}{\partial\eta} \to 0 \quad \text{as} \quad \eta \to \infty.\end{aligned}\right\} \tag{4.15}$$

The solution is

$$\begin{aligned}u_{16} = &-\frac{53M^4 - 10iM^2 - 3}{2(M^3 - iM)^2}\exp\left[-(\sqrt{i})\eta\right] - \frac{1}{(M^3 - iM)}\left[\frac{1}{3}\eta^3 + \frac{1}{2M}\eta^2\right.\\&+ \frac{1}{2M^2}\eta + \frac{1}{M^3 - iM}\left\{\frac{M^2 - i}{2M^2} - \frac{3M^2 + i}{M}\eta + (3M^2 - i)\eta^2\right\}\\&\left.- \frac{1}{(M^3 - iM)^2}\left\{\frac{(3M^2 - i)(9M^2 + i)}{M} - 2(3M^2 - i)^2\eta\right\}\right]\exp(-M\eta),\end{aligned} \tag{4.16}$$

where $M = \sqrt{(i\sigma)}$.

When the frequency of oscillation is large the temperature field may be written as

$$T = T_s + \epsilon R_5 \exp[-(\sqrt{\sigma}/2)\,\eta] \cos[\omega t - (\sqrt{\sigma}/2)\,\eta - \alpha_5], \tag{4.17}$$

where

$$R_5 = (P_5^2 + Q_5^2)^{\frac{1}{2}}, \qquad \alpha_5 = \tan^{-1}(Q_5/P_5),$$

$$P_5 = 1 + \frac{3\sigma N^2 x^{-\frac{6}{5}}}{250\,\omega^{\frac{3}{2}}}\left(\frac{\eta^3}{3} + \frac{\eta^2}{2(\sqrt{2\sigma})}\right),$$

$$Q_5 = \frac{3\sigma N^2 x^{-\frac{6}{5}}}{250\,\omega^{\frac{3}{2}}}\left(\frac{\eta^2}{2(\sqrt{2\sigma})} + \frac{\eta}{2\sigma}\right).$$

The local Nusselt number from the surface to the fluid for high frequencies may be written as

$$Nu = -\left(\frac{\partial T_s}{\partial y}\right)_{y=0} - \epsilon \exp(i\omega t)\left(\frac{\partial T_1}{\partial y}\right)_{y=0}$$

$$= N^{-1} x^{-\frac{2}{5}} + \epsilon R_6 \cos(\omega t + \alpha_6), \tag{4.18}$$

where

$$R_6 = \left\{\frac{\sigma\omega}{2} + \left[\sqrt{\left(\frac{\sigma\omega}{2}\right)} + \frac{3N^2 x^{-\frac{6}{5}}}{500\,\omega}\right]^2\right\}^{\frac{1}{2}},$$

$$\alpha_6 = \tan^{-1}\left(1 + \frac{3(\sqrt{2})\,N^2 x^{-\frac{6}{5}}}{500\,(\sqrt{\sigma})\,\omega^{\frac{3}{2}}}\right).$$

The velocity gradient at the plate can be written as

$$Re\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\partial u_s}{\partial y}\right)_{y=0} - \epsilon\,\omega^{-\frac{5}{2}} R_7 \cos\left(\omega t - \frac{\pi}{4}\right), \tag{4.19}$$

where

$$R_7 = \left[(\sqrt{\sigma}) - 1\right]\left[\frac{53\sigma^2 - 10\sigma - 3}{2\sigma^2(1-\sigma)^3}\right] + \left[\frac{31\sigma^2 - 22\sigma + 7}{2\sigma^{\frac{3}{2}}(1-\sigma)^3}\right].$$

It is interesting to note that for very high frequencies, the velocities set up will be extremely small being of the order of ω^{-3}. The Nusselt number at the plate has a phase lead which tends to shear wave value $\pi/4$ as $\omega \to \infty$, while the skin-friction has a phase lag of $\pi/4$. These results are the same as obtained by Nanda and Sharma [4] for a vertical plate.

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REFERENCES

1. W. N. Gill and E. D. Casal, A theoretical investigation of natural convection effects in forced horizontal flows, *A.I.Ch.E. Jl* **8**, 513–518 (1962).
2. E. M. Sparrow and W. J. Minkowycz, Buoyancy effects on horizontal boundary-layer flow and heat transfer, *Int. J. Heat Mass Transfer* **5**, 505–511 (1962).
3. M. J. Lighthill, The response of laminar skin-friction and heat transfer to fluctuations in the stream velocity, *Proc. R. Soc.* **A 224**, 1 (1954).
4. R. S. Nanda and V. P. Sharma, Free convection laminar boundary layers in oscillatory flow, *J. Fluid Mech.* **15**, 419–428 (1963).

Résumé—La convection naturelle transitoire à partir d'une plaque horizontale semi-infinie est analysée lorsque la température de la plaque varie périodiquement dans le temps autour d'une valeur moyenne constante. On établit des solutions séparées pour les gammes de fréquences basses et élevées. On trouve que la valeur quasi-permanente ($\omega = 0$) du nombre de Nusselt sur la plaque croît lorsque le nombre de Prandtl augmente. Pour des fréquences très élevées le champ de température est du type onde de cisaillement non modifiée par l'écoulement moyen permanent.

Zusammenfassung—Instationäre freie Konvektion von einer halbunendlichen waagerechten Platte wird wird für den Fall analysiert, dass sich die Plattentemperatur periodisch um einen konstanten Mittelwert ändert. Getrennte Lösungen sind für niedrige und hohe Frequenzen ermittelt. Es zeigt sich, dass für niedrige Frequenzen die Schwingungskomponente der Nusseltzahl an der Platte zunimmt, bei zunehmender Prandtlzahl. Für sehr hohe Frequenzen zeight das Temperaturfeld Scherwellen und bleibt unbeeinflusst vom mittleren stationären Fluss.

Аннотация—Проведен анализ процесса нестационарной свободной конвекции полубесконечной горизонтальной пластины при периодическом изменении со временем ее температуры. Получены отдельные решения для диапазонов высоких и низких частот. Найдено, что с ростом числа Прандтля увеличивается квазистационарное значение ($\omega = 0$) критерия Нуссельта. При очень больших частотах поле температуры сходно с волной касательного напряжения, на зависящей от стационарного потока.